Lecture 9

Monday, February 3, 2020 6:19 PM

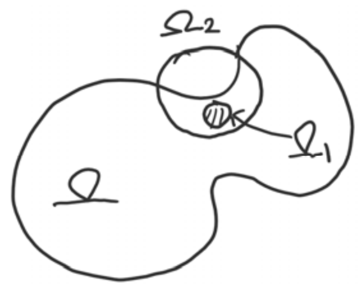
- Finish discussion of Reinhardt domains and log convex hulls from Lecture 8.

Domains of holomorphy.

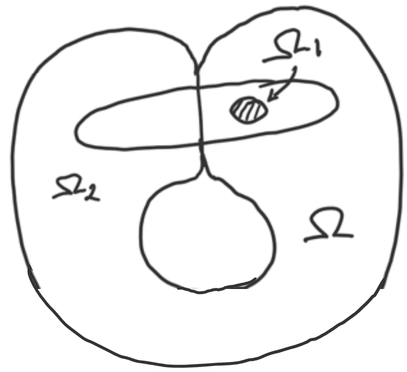
Def 2. A domain $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy if $\nexists\ \Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ s.t.

(i) $\emptyset \neq \Omega_1 \subseteq \Omega_2 \cap \Omega$

(ii) $\forall u$ holom. in Ω $\exists U$ holom. in Ω_2 s.t. $U|_{\Omega_1} = u$.



In this picture, all u holom. in Ω would extend to a holom. U in $\Omega \cup \Omega_2$, but:



in this picture, $\Omega_2 \subseteq \Omega$ so holom. fcns in Ω do not need to extend to a larger domain, but they all extend locally across $\partial\Omega$. (like $\log z$ in the slit plane).

- A main goal is to characterize domains of holomorphy by local conditions on the boundary $\partial\Omega$.
- We shall introduce the notation $\mathcal{O}(\Omega)$ for the space of holomorphic functions in a domain Ω.

<u>Def. 1.</u> Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $K \subset\subset \Omega$ cpt. The $\mathcal{O}(\Omega)$-hull of K, $\hat{K}_\Omega$, is defined by

$$\hat{K}_\Omega := \left\{ z \in \Omega : |f(z)| \le \sup_{\zeta \in K} |f(\zeta)|, \ \forall f \in \mathcal{O}(\Omega) \right\}.$$

- $\hat{K}_\Omega$ is closed in Ω (but could have limit pts on $\partial\Omega$, so not closed in $\mathbb{C}^n$).
- <u>Ex.</u> The <u>convex hull</u> of K (in $\mathbb{C}^n$) can be described as follows:

$$CH(K) = \left\{ z \in \mathbb{C}^n : |f(z)| \le \sup_K |f|, \ f(z) = e^{z \cdot \zeta}, \ \zeta \in \mathbb{C}^n \right\}$$

$$z \cdot \zeta = \sum_j z_j \zeta_j$$

It follows that $\hat{K}_\Omega \subseteq CH(K) \Rightarrow \hat{K}_\Omega$ is bdd.

<u>Thm 1.</u> Let $\Omega \subseteq \mathbb{C}^n$. TFAE:

(i) Ω is a domain of holomorphy.

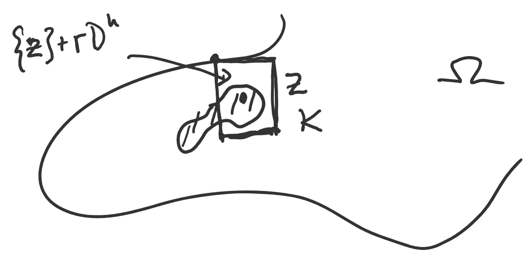
(ii) If $K \subset\subset \Omega$, then $\hat{K}_\Omega \subset\subset \Omega$.

(iii) $\exists f \in \mathcal{O}(\Omega)$ s.t. f does not extend across any boundary pt.
I.e., $\not\exists$ $\Omega_2 \not\subseteq \Omega$, $\underset{\neq \emptyset}{\Omega_1} \subseteq \Omega \cap \Omega_2$, $F \in \mathcal{O}(\Omega_2)$ s.t. $F|_{\Omega_1} = f$.

We shall need a lemma for the pf of Thm 1.

Let D^n be a fixed polydisk centered at $a = 0$; $D^n = \{ z : |z_j| < r_j, \ j = 1, \dots, n \}$. $r = (r_1, \dots, r_n)$

Let $K \subset\subset \Omega$ cpt, and $\Delta(z) = \Delta_{\Omega, D^n}(z) = \sup\{ \rho > 0 : \{z\} + \rho D^n \subseteq \Omega \}$



Lemma 1. Let $f \in \mathcal{O}(\Omega)$ satisfies $|f(z)| \le \Delta(z)$, $z \in K$. If $u \in \mathcal{O}(\Omega)$, then $\forall\, \zeta \in \hat{K}_\Omega$

$$u(z) = \sum_\alpha \frac{u_{z^\alpha}(\zeta)}{\alpha!}(z-\zeta)^\alpha$$

w/ normal convergence in the polydisk $\{\zeta\} + |f(\zeta)|\, D^n$.

Pf: Fix $0<s<1$, consider $K' = \{z \in \Omega : \{w\} + s|f(w)|\overline{D^n}, w \in K\} = \{z \in \Omega : |z_j - w_j| \le s|f(w)| r_j,\ j = 1, \dots, n,\ w \in K\}$. Since $s<1$, K' is compact (Ex.) $\Rightarrow |u(z)| \le M$, $z \in K'$.

For $z \in K$, since $\{z\} + s|f(z)|\, D^n \subseteq \Omega$, Cauchy's Estimates $\Rightarrow$

$$|u_{z^\alpha}(z)| \le \frac{M\, \alpha!}{(s|f(z)|)^{|\alpha|} \cdot r^\alpha} \iff \left|\frac{u_{z^\alpha}(z)\, f(z)^{|\alpha|}}{\alpha!}\right| \le \frac{M}{s^{|\alpha|} r^\alpha}. \quad (1)$$

Each $u_{z^\alpha}(z)\, f(z)^{|\alpha|} \in \mathcal{O}(\Omega) \Rightarrow (1)$ holds also on $\hat{K}_\Omega$, i.e., if $\zeta \in \hat{K}_\Omega$ we have

$$\frac{|u_{z^\alpha}(\zeta)|}{\alpha!} \le \frac{M}{(s|f(\zeta)|)^{|\alpha|}\, r^\alpha},$$

which implies that $\sum_\alpha \frac{u_{z^\alpha}(\zeta)}{\alpha!}(z-\zeta)^\alpha$ converges normally (to $u(z)$) in the polydisk $\{\zeta\} + s|f(\zeta)|\, D^n$.

normally (to $u(z)$) in the polydisk $\{\zeta\} + s|f(\zeta)|D$.
Since $s<1$ was arbitrary, the conclusion of Lemma 1 follows. ☒

- Let $\delta(z) = \max_j |z_j|$ (L^∞-norm), and $\delta(\underset{\text{compact}}{K}, \underset{\text{closed}}{F}) = \inf_{z\in K, w\in F} \delta(z-w)$, i.e., distance $d_\mu(K,F)$ measured using metric $\delta(z-w)$.
- This metric on $\mathbb{C}^n$ is good for power series b/c the "balls" $B(a,\rho) = \{z\in\mathbb{C}^n : \delta(z-a)<\rho\}$ are polydisks centered at a w/ poly radius $(\rho,\dots,\rho)$.

<u>Cor 1</u>. Let $\Omega \subseteq \mathbb{C}^n$ be domain of holomorphy. If $f\in\mathcal{O}(\Omega)$ and $|f(z)| \le \delta(z, \mathbb{C}^n\setminus\Omega)$ $\forall z\in K$, then
$$|f(\zeta)| \le \delta(\zeta, \mathbb{C}^n\setminus\Omega), \ \forall \zeta\in\hat{K}_\Omega. \quad (2)$$

In particular, $$\delta(K, \mathbb{C}^n\setminus\Omega) = \delta(\hat{K}_\Omega, \mathbb{C}^n\setminus\Omega). \ (3)$$

<u>Pf</u>. Let D^n be polydisk $\{z\in\mathbb{C}^n : \delta(z)<1\}$ in Lemma 1. Then, $\Delta(z) = \delta(z, \mathbb{C}^n\setminus\Omega)$, so Lemma 1 implies that $\forall \zeta\in\hat{K}_\Omega$, $u\in\mathcal{O}(\Omega)$, the power series of u converges normally in $\{\zeta\}+|f(\zeta)|D^n$.
If, for some $\zeta\in\hat{K}_\Omega$, (2) does not hold, then $\Omega_2 = \{\zeta\}+|f(\zeta)|D^n$ is <u>not</u> contained in Ω and hence $\sum_\alpha \frac{u_\alpha(\zeta)}{\alpha!}(z-\zeta)^\alpha$ is holom. in Ω_2, and coincides w/ $u(z)$ in $\Omega_1 = \Omega\cap\Omega_2$. This contradicts Ω being a domain of holomorphy.
Since $\delta(K, \mathbb{C}^n\setminus K) = \varepsilon > 0$, we may use $f(z)\equiv\varepsilon$. (2) $\Rightarrow$ $\delta(K, \mathbb{C}^n\setminus\Omega) \le \delta(\hat{K}_\Omega, \mathbb{C}^n\setminus\Omega)$. Since $K\subseteq\hat{K}_\Omega \Rightarrow$ (3). ☒

$\delta(K, \mathbb{C}^n \setminus \Omega) \leq \delta(\hat{K}_\Omega, \mathbb{C}^n \setminus \Omega)$. Since $K \subseteq \hat{K}_\Omega \Rightarrow (3)$. $\blacksquare$

Cor 2. If $\Omega \subseteq \mathbb{C}^n$ is domain of holomorphy, $K \subset\subset \Omega$, then $\hat{K}_\Omega \subset\subset \Omega$.

Pf. $\hat{K}_\Omega$ is clearly closed in Ω. Compactness follows from (3) in Cor 1, since $\delta(\hat{K}_\Omega, \mathbb{C}^n \setminus \Omega) = \delta(K, \mathbb{C}^n \setminus \Omega) > 0$. $\blacksquare$